

# Loday constructions for Tambara functors

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joint work with Ayelet Lindenstrauss and Foling Zou

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- ▶ for every pair of finite  $G$ -sets  $X$  and  $Y$ , applying  $M_*$  to  $X \rightarrow X \sqcup Y \leftarrow Y$  gives the component maps of an isomorphism  $\underline{M}(X) \oplus \underline{M}(Y) \cong \underline{M}(X \sqcup Y)$ .

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$\underline{A}$  is initial in  $\text{Tamb}_G$  and a unit for the so-called box product of  $G$ -Mackey functors,  $\square$ .

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2. There is a natural isomorphism  $X \otimes (Y \otimes \underline{R}) \cong (X \times Y) \otimes \underline{R}$ .
3. On the category with objects finite sets with trivial  $G$ -action and morphisms consisting only of isomorphisms, the functor restricts to exponentiation  $X \otimes \underline{R} = \prod_{x \in X} \underline{R}$ .

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The pair  $(N_H^G, i_H^*)$  is an adjoint functor pair.

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The proof is by direct inspection, where we use the fact that  $\underline{R}^c \square \underline{R}^c \cong \underline{(R \otimes R)^c}$ .

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The next result is a fun fact about fixed points: Let  $G = C_2$  and let  $X$  be any finite  $C_2$ -simplicial set.

**Proposition**[The hungry fixed points]

$$\mathcal{L}_X^{C_2}(\underline{\mathbb{Z}}^c) \cong \begin{cases} \underline{\mathbb{Z}}^c, & \text{if } X^{C_2} \neq \emptyset, \\ \underline{A}, & \text{if } X^{C_2} = \emptyset. \end{cases}$$

We saw that  $\underline{A}$  is the initial object in  $\text{Tamb}_{C_2}$  and the ring of integers is initial in the category of commutative rings. Therefore

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**The twisted cyclic nerve**

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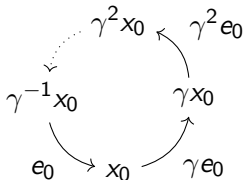
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We have  $(S_{\text{rot}}^1)_k = \{C_n \cdot x_k^0, C_n \cdot x_k^1, \dots, C_n \cdot x_k^k\}$ , where

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The simplicial identities imply that

$$\begin{aligned} d_j(x_k^0) &= x_{k-1}^0, \\ d_j(x_k^i) &= \begin{cases} x_{k-1}^{i-1} & 0 \leq j \leq i-1 \\ x_{k-1}^i & i \leq j \leq k \text{ and } i \neq k \end{cases} \\ d_k(x_k^k) &= \gamma^{-1} x_{k-1}^0. \end{aligned}$$

So for a  $C_n$ -Tambara functor  $\underline{R}$  with  $R := i_e^* \underline{R}$ , there is

$$\mathcal{L}_{S_{\text{rot}}^1}^{C_n}(\underline{R})_k = \square_{0 \leq i \leq k} (C_n \otimes \underline{R}) = (N_e^{C_n} R)^{\square(k+1)},$$

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For every subgroup  $K < C_n$  we can identify the twisted cyclic nerve relative to  $K$  as

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In particular, for  $K = C_n$ :

$$\mathcal{L}_{S_{\mathrm{rot}}^1/C_n}^{C_n}(\underline{R}) \cong \underline{\mathrm{HC}}_{C_n}^{C_n}(\underline{R}) = \underline{\mathrm{HC}}^{C_n}(\underline{R}).$$